References

[1] Goulard, R., "The coupling of radiation and convection in detached shock layers," Bendix Product Div., Applied Sciences Lab., RN 23 (December 1960).

[2] Wick, B. H., "Radiative heating of vehicles entering the earth's atmosphere," Fluid Mechanics Panel, AGARD, Brussels, Belgium (April 1962).

[3] Yoshikawa, K. K. and Chapman, D. R., "Radiative heat transfer and absorption behind a hypersonic normal shock wave," NASA TN D-1424 (September 1962).

[4] Howe, J. T. and Viegas, J. R., "Solutions of the ionized radiating shock layer, including reabsorption and foreign species effects, and stagnation region heat transfer," NASA TR R-159 (1963).

[5] Wilson, K. H. and Hoshizaki, H., "Inviscid, nonadiabatic flow about blunt bodies," AIAA Preprint 64-70 (January 1964).

[6] Hayes, W. D. and Probstein, R. F., *Hypersonic Flow Theory* (Academic Press, New York, 1959).

[7] Viegas, J. R. and Howe, J. T., "Thermodynamic and transport property correlation formulas for equilibrium air from 1,000 K to 15,000 K," NASA TN D-1429 (October 1962).

Electron Fluctuations in an Equilibrium Turbulent Plasma

ANTHONY DEMETRIADES*
Philco Corporate Research Laboratories, Newport Beach, Calif.

I. Introduction

THE radar cross section of an underdense turbulent wake depends on the mean square of the electron density fluctuations and on their spectral distribution.[1] Since no experiment indicating the fluctuation magnitude is available at this writing, one is forced to resort to empirical descriptions of the turbulent mixing process. One such empirical model, variously known as the "marble-cake" or "inviscid convection" model, has been elaborated upon by Feldman and Proudian[2] and by Lin and Hayes.[3] A detailed criticism of this pseudoturbulent process is permitted here neither by the limited space nor by our long-lacking understanding of the problem even when ionization is absent.

Instead of attempting to predict the magnitude of the gasdynamical (temperature, density, etc.) fluctuations, we will attempt to simply derive formulas connecting these gasdynamical fluctuations to electron density fluctuations in the extreme of equilibrium flow. Here, the "equilibrium" terminology refers to the microscopic processes enacted within the details of the turbulent structure. In this manner, we limit the present formulation to the case where the local electron density at a point is determined only by the gas density and temperature at that point.

Implicit in the following work is the assumption that, although the turbulence affects the electron population through density or temperature fluctuations, the electrical, diffusive, or inertial effects of the electrons and ions do not affect turbulent mixing as presently, albeit sketchily, understood. In this way, the results of the present paper can be applied to experimental findings on gasdynamical fluctuations alone, and the corresponding electron fluctuation magnitude can be inferred from those. Another purpose of the present formulation is to provide relations useful to any type of turbulent plasma rather than an ionized wake alone, without specifying the mixing model as in the case of Feldman[2] and Lin.[3]

II. Interdependence of Gasdynamical Fluctuations

Since electron density fluctuations are related to the gasdynamical fluctuations, we discuss first the possibility of describing the turbulent field by means of the fluctuations of only one dynamic or thermodynamic variable.

Many years ago Kovasznay[4] pointed out that the gasdynamical fluctuations in a turbulent gas can be uniquely decomposed into three linearly independent fluctuation "modes": entropy, sound, and vorticity. In each of these modes there exists a unique relationship among pressure, density, temperature, and velocity fluctuations as dictated by the equations of conservation. In the so-called entropy mode, the temperature T and density ρ fluctuations are related by

$$\Delta\rho/\bar{\rho} = -(\Delta T/\bar{T}) \tag{1}$$

where bars denote averages and Δ stands for the fluctuation. In the "sound" mode, the corresponding relationship is

$$\Delta\rho/\bar{\rho} = [1/(\gamma - 1)](\Delta T/\bar{T}) \tag{2}$$

where γ is the ratio of specific heats. The "vorticity" mode involves fluctuations in the fluid velocity only and is, therefore, of no consequence as regards the electron density.

Physical considerations usually dictate the fluctuation mode (sound or entropy) predominant to a given situation.† Working with turbulent supersonic boundary layers, Kovasznay[4] and later Kistler[5] found that, in the absence of considerable sound radiation, the sound mode seems negligible compared to the entropy mode. In a turbulent hypersonic wake, such "sound generators" often are the shocklet-capped "turbs" found in the near wake, whose mean speed is considerably lower than that of the external flow; in the far wake, by contrast, the wake has acquired the velocity of the external stream and "sound" production is negligible, and the persisting wake temperatures probably make the vorticity and entropy modes dominant. Insofar as the electron fluctuations are concerned, the latter mode would then be the only one of significance.

III. Average Electron Density

At equilibrium, the instantaneous electron number density n_e is given in terms of the gas number density n and temperature T by the Saha relation

$$n_e^2 = AnT^{3/2}e^{-(B/T)} \tag{3}$$

where A is a constant. The quantity $B \equiv E_i/k$, where E_i is the ionization energy and k the Boltzmann constant.

Suppose that the time dependence of the turbulent gas temperature can be written as

$$T(t) = \bar{T} + \Delta T(t) \tag{4}$$

where the time average $\overline{\Delta T}$ is zero by definition, so that $\bar{T}$ is the average (mean) gas temperature. From (3), we get

$$n_e^2(\bar{T}) = A\bar{n}\bar{T}^{3/2}e^{-(B/\bar{T})} \tag{5}$$

We will first show that $n_e(\bar{T})$ is not, in fact, the average electron density which obtains as the electron density fluctuates.

We show this as follows: if $n_e(T)$ is expanded around $\bar{T}$, we obtain

$$n_e(T) = n_e(\bar{T}) + \left(\frac{\partial n_e}{\partial T}\right)_{\bar{T}} \Delta T + \frac{1}{2}\left(\frac{\partial^2 n_e}{\partial T^2}\right)_{\bar{T}} (\Delta T)^2 + \ldots + \frac{1}{m!}\left(\frac{\partial^m n_e}{\partial T^m}\right)_{\bar{T}} (\Delta T)^m + \left(\frac{\partial n_e}{\partial \rho}\right)_{\bar{\rho}} \Delta\rho + \ldots \tag{6}$$

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* Principal Scientist, Fluid Mechanics Research Department. Associate Fellow Member AIAA.

† In the entropy mode, temperature and density fluctuations are perfectly anticorrelated [Eq. (1)], and in the sound mode, they are perfectly correlated [Eq. (2)]. If feasible, a correlation experiment could therefore resolve this point.

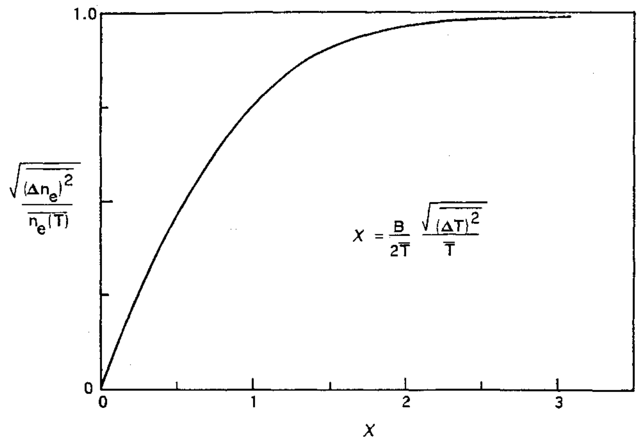

Fig. 1 Dependence of the fractional root-mean-square electron density fluctuations on the virtual gas temperature fluctuations in the case of local equilibrium.

Now the derivatives can be computed from (5):

$$\left(\frac{\partial^m n_e}{\partial T^m}\right)_{\bar{T}} = A^{1/2}\bar{n}^{1/2}\bar{T}^{(3/4)-m}\left[\left(\frac{B}{2\bar{T}}\right)^m + O\left\{\left(\frac{B}{2\bar{T}}\right)^{m-1}\cdots\right\}\right]$$

and since $B/2\bar{T} \gg 1$, we can retain the leading term only. If we divide by $n_e(\bar{T})$, we obtain

$$\frac{1}{n_e(\bar{T})}\left(\frac{\partial n_e}{\partial T}\right)_T \Delta T \simeq \left(\frac{B}{2\bar{T}}\right)\frac{\Delta T}{\bar{T}} \tag{7}$$

and

$$\frac{1}{n_e(\bar{T})}\left(\frac{\partial^m n_e}{\partial T^m}\right)_{\bar{T}} \simeq \left(\frac{B}{2\bar{T}}\right)^m \frac{1}{\bar{T}^m} \tag{8}$$

If we invoke, on the other hand, Eq. (5) and either relation (1) or (2), we obtain

$$\frac{1}{n_e(\bar{T})}\left(\frac{\partial n_e}{\partial \rho}\right)_{\bar{\rho}} \Delta\rho = \frac{1}{2}\frac{\Delta\rho}{\rho} \simeq \frac{\Delta T}{\bar{T}} \tag{9}$$

and, by comparing (9) with (7), we see that the density derivatives in the expansion for $n_e\ (T)$ can be neglected in favor of the temperature derivatives (because $B/2\bar{T} >> 1$). On taking the time average, Eq. (6) therefore becomes

$$\overline{n_e(T)} = n_e(\bar{T}) + \frac{1}{2}\left(\frac{\partial^2 n_e}{\partial T^2}\right)_{\bar{T}}\overline{(\Delta T)^2} + \ldots + \frac{1}{m!}\left(\frac{\partial^m n_e}{\partial T^m}\right)_{\bar{T}}\overline{(\Delta T)^m} \tag{10}$$

and, if we insert (8) into (10), we obtain

$$\frac{\overline{n_e(T)}}{n_e(\bar{T})} = 1 + \sum_m \frac{1}{m!}\left(\frac{B}{2\bar{T}}\right)^m \frac{\overline{(\Delta T^m)}}{\bar{T}^m} \qquad m = 2, 4, 6 \ldots \tag{11}$$

The terms $\overline{(\Delta T)^6}$, $\overline{(\Delta T)^8}$, etc., cannot be reduced in terms of, say, $\overline{(\Delta T)^2}$. However, it is certainly reasonable to approximate‡ these by

$$\overline{(\Delta T)^m} = [\overline{(\Delta T)^2}]^{m/2} \tag{12}$$

which reduces (11) to

$$\frac{\overline{n_e(T)}}{n_e(\bar{T})} = 1 + \sum_m \frac{1}{m!}\left[\left(\frac{B}{2\bar{T}}\right)\frac{[\overline{(\Delta T)^2}]^{1/2}}{\bar{T}}\right]^m \qquad m = 2, 4, 6 \ldots \tag{13}$$

‡ For a square wave, for example, $[\overline{(\Delta T)^2}]^2 = \overline{(\Delta T)^4}$ exactly; for a sine wave, $[\overline{(\Delta T)^2}]^2 = \frac{2}{3}\overline{(\Delta T)^4}$. The assumption, however, remains extremely crucial. If the $\overline{(\Delta T^m)}$ become available from experiments, equations such as (11) should be used in place of (16).

and if the "virtual fluctuation" in the bracket is written as

$$\chi = \frac{B}{2\bar{T}}\frac{[\overline{(\Delta T)^2}]^{1/2}}{\bar{T}} \tag{14}$$

then

$$\frac{\overline{n_e(T)}}{n_e(\bar{T})} = 1 + \sum_m \frac{1}{m!}\chi^m \qquad m = 2, 4, 6 \ldots \tag{15}$$

or

$$\frac{\overline{n_e(T)}}{n_e(\bar{T})} = \cosh\chi \tag{16}$$

This remarkably simple formula shows that the "proper" average electron density can be vastly different from the pseudoaverage $n_e(\bar{T})$ density corresponding to the mean temperature. When the plasma is very hot and the fluctuation level is low, χ is small, and the two averages are about the same. When, however, the plasma is relatively cold and the turbulent intensity is high, the correction becomes extremely large because χ is large. Consider, for instance, that at $\bar{T}$ = 3000° (for air) $B/2\bar{T}$ is about 29. If the root-mean-square temperature fluctuation is 10% of the mean, χ is about 3, and if the fluctuation is 35% of the mean, χ is about 10.§ We therefore conclude from Eq. (16) that, if one wrongly evaluates the electron density at the mean temperature [thus obtaining the pseudoaverage $n_e(\bar{T})$], one may underestimate the proper average electron density by orders of magnitude.

IV. Fluctuations

We next calculate the root-mean-square electron density fluctuation at equilibrium, defined by

$$[\overline{(\Delta n_e)^2}]^{1/2} = \{\overline{[n_e(T) - \overline{n_e(T)}]^2}\}^{1/2} \tag{17}$$

If Eqs. (6) (with the density derivatives deleted) and (10) are inserted into (15) and the necessary operations performed, we obtain $\overline{(\Delta n_e)^2}$ as another power series in χ:

$$\overline{(\Delta n_e)^2}/n_e(\bar{T})^2 = x^2 + 0.33\chi^4 + 0.0445\chi^6 + 3.2 \times 10^{-3}\chi^8 + 1.41 \times 10^{-4}\chi^{10} + \ldots = \sinh^2\chi \tag{18}$$

where the approximation (12) has again been used. The simple hyperbolic sine form is a direct result of this approximation.

The mean-square fractional electron density fluctuation can now be calculated from (16) and (18):

$$\frac{[\overline{(\Delta n_e)^2}]^{1/2}}{\overline{n_e(T)}} = \frac{[\overline{(\Delta n_e)^2}]^{1/2}}{n_e(\bar{T})}\frac{n_e(\bar{T})}{\overline{n_e(T)}} = \frac{1}{\cosh\chi}\sinh\chi = \tanh\chi \tag{19}$$

with the result shown on Fig. 1, where one sees that the fluctuation magnitude approaches unity soon after $\chi > 1$. This graph, therefore, allows one to conclude that at equilibrium the root-mean-square electron fluctuation is indeed equal to the (proper) average electron density, regardless of the fluctuation mode and magnitude. Exceptions to this rule would occur only when the root-mean-square temperature fluctuations decrease to low values (below about 4% for $\bar{T}$ = 3000°K).

V. Conclusions

In the foregoing, we have derived formulas converting static temperature fluctuations in a heated gas to fluctuations in the electron density in the case of equilibrium. We have shown that the numerical equality of the root-mean-square electron density fluctuation to the proper average electron density is indeed very likely. Moreover, this average electron

§ Demetriades[6] gives fluctuation values of order 10 to 40% of the mean values in a hypersonic wake. Slattery[7] claims density fluctuations as high as 90% of the mean.

density is usually much higher than the electron density computed at the mean local temperature. The importance of this in interpreting results of plasma experiments is all too obvious.

The question of whether the turbulent flow is in an equilibrium or frozen state remains important. This can be settled only if a comparison can be made of the relevant chemical relaxation times with the residence times of a fluid particle in a given "turb." Such a comparison cannot be made accurate within the present understanding of turbulent mixing. However, the observed importance of the entropy mode of fluctuations implies residence times large enough to be comparable with the flow times, such as the transit time between the body and the relevant station in the wake, for example; the probability of local equilibration is thus enhanced.

References

[1] Menkes, J., "Scattering of radar waves by an under dense turbulent plasma," AIAA Preprint 64-20 (January 20, 1964).

[2] Feldman, S. and Proudian, A., "Some theoretical predictions of mass and electron density oscillations based on a single model for turbulent wake mixing," AIAA Preprint 64-21 (January 20, 1964).

[3] Lin, S. C. and Hayes, J. E., "A quasi-one-dimensional model for chemically reacting turbulent wakes of hypersonic objects," Avco Everett Research Lab., RR 157 (July 1963).

[4] Kovasznay, L. S. G., "Turbulence in supersonic flow," J. Aeronaut. Sci. **20**, 657 (1953).

[5] Kistler, A., "Fluctuation measurements in a supersonic turbulent boundary layer," Phys. Fluids **3**, 290 (1959).

[6] Demetriades, A., "Some hot-wire anemometer measurements in a hypersonic wake," *Proceedings of the 1961 Heat Transfer and Fluid Mechanics Institute* (Stanford University Press, Stanford, Calif., 1961), p. 1.

[7] Slattery, R. E. and Clay, W. G., "The turbulent wake of hypersonic bodies," ARS Paper 2673-63 (November 1963).

Least-Squares Approach to Unsteady Kernel Function Aerodynamics

JOSEPH A. FROMME*
North American Aviation, Columbus, Ohio

A LEAST-SQUARES solution to the integral equation relating pressure and downwash in three-dimensional unsteady flow has distinct advantages over collocation solutions, particularly with regard to alleviating error between collocation points. Although the mathematical aspects of least-squares approximation to complex valued functions in higher dimensions are not well known,[1] the results are simple and straightforwardly applied. In view of this simplicity of application, it is hoped that the method employed herein will be useful to a wide variety of other engineering problems.

Basic Equations and Motivation

The problem of determining unsteady aerodynamic forces on finite span planforms leads to a singular homogeneous Fredholm integral equation between pressure and downwash.[2] Letting C_p be the perturbation pressure coefficient and w the downwash nondimensionalized by freestream velocity, the integral equation is

$$w(x, y) = \frac{1}{8\pi} \iint_s C_p(\xi, \eta) K(x - \xi, y - \eta; M, k) d\xi d\eta \quad (1)$$

where S is the planform area, M and k are the Mach number and nondimensional reduced frequency, x, ξ and y, η are nondimensional chordwise and spanwise variables, and K is the complex valued kernel function. Workable expressions for K in subsonic[2] and supersonic[3] flow were given by Watkins et al. of NASA. Successful digital computer programs for predicting three-dimensional unsteady airloads based on an assumed mode approach were originally developed notably by Watkins, et al.[4, 5] of NASA and Hsu, et al.[6] of Massachusetts Institute of Technology (MIT). Assuming that the pressure may be represented as a superposition of preselected pressure modes,

$$C_p(x, y) = \sum_i p_i(x, y) b_i \quad (2)$$

the coefficients b_i must satisfy the equation

$$w(x, y) = \sum_i \left[\iint_s p_i(\xi, \eta) K(x - \xi, y - \eta; M, k) d\xi d\eta\right] b_i \quad (3)$$

Equation (3) cannot, in general, be satisfied at all points on the planform. The forementioned computer programs require that Eq. (3) be satisfied at as many points as there are assumed modes; i.e., "downwash collocation." The resulting set of algebraic equations constitute a complex matrix equation

$$\{w(x_i, y_i)\} = [D_{ij}]\{b_i\} \quad (4)$$

where the matrix definitions are clear from Eq. (3).

An attempt was made to reduce error in the MIT program. The flow is regarded as two-dimensional, and collocation stations are chosen so that the error is zero *on the average*.[6] However, the flow is three-dimensional, particularly with modern low-aspect ratio surfaces. In addition, because "zero error on the average" means that equally bad errors of opposite sign are equivalent to two points of zero error, it is desirable to minimize the mean-square error over the entire surface.

Equation (4) guarantees only that the downwash is correct at collocation points; no guarantee is made between these points. Error between collocation points can seriously affect the solution, although no serious difficulties have occurred for all-movable flexible surfaces. However, there have been instances in supersonic flow where the airloads, due to the first and second vibration modes of surfaces cantilevered at the root, could not be calculated because of excessive error between collocation points. In addition, regardless of the surface and flow conditions, the solutions depend to some extent on collocation point locations, thereby creating a recurring question with regard to proper location of the points. Furthermore, it is sometimes desirable to leave out certain higher-order assumed pressure modes. However, the collocation solution necessitates an equal decrease in the number of collocation points, thereby increasing error. These difficulties would all be alleviated by a least-squares solution to Eq. (3). Moreover, it can be shown[1] that, in the least-squares case, ill conditioning is a direct measure of linear dependence of approximating functions, thereby providing, for the first time, a *usable* criterion for the adequacy of assumed pressure modes. This is particularly useful in supersonic flow where ill conditioning has been a problem.

The Least-Squares Approach

For notational simplicity, denote the integrals within brackets in Eq. (3) by $w_i(x, y)$. Then the problem becomes that of approximating the known downwash $w(x, y)$ by the downwash functions induced by the assumed pressure modes:

$$w(x, y) = \sum_1^n w_i(x, y) b_i \quad (5)$$

where n is the number of assumed modes. It is shown in Ref. 1 that the least-square error over the surface is obtained

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* Senior Dynamics Engineer, Research and Development Department; also Graduate Student, Department of Engineering Mechanics, The Ohio State University, Columbus, Ohio.